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Single Server Queues with Restricted Accessibility

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SUMMARY
A queueing system has restricted accessibility if not every customer is admitted to the system. For such a system the admittance of a customer will in general depend on the state of the queueing system at the moment of his arrival. In this paper queueing models will be studied for which the accessibility depends on the actual waiting time of the arriving customer. Various queueing situations encountered in the allocation of memory equipment for information processing systems may be described as a single server queueing system with restricted accessibility; the mathematical models for the involved storage problems belong to queueing theory and are discussed in the present paper.

1. Introduction

A queueing system has restricted accessibility if not every customer is admitted to the system. For such a system the admittance of a customer will in general depend on the state of the queueing system at the moment of his arrival. A system with a limited number of waiting places is a typical example of restricted accessibility. An arriving customer who finds all waiting places occupied is not admitted (cf. Cohen [1]).

In this paper we shall consider several single server queueing models with accessibility depending on the actual waiting time of the arriving customer.

Let $\boldsymbol{\sigma}_n$, $n=1, 2, \ldots$, denote a sequence of independent, nonnegative and identically distributed variables with distribution function $A(\cdot)$; $\boldsymbol{\tau}_n$, $n=1, 2, \ldots$, is another sequence of independent, nonnegative and identically distributed variables with distribution $B(\cdot)$. It will be assumed that the families $\{\boldsymbol{\sigma}_n, n=1, 2, \ldots\}$ and $\{\boldsymbol{\tau}_n, n=1, 2, \ldots\}$ are independent families of stochastic variables. $\boldsymbol{\sigma}_{n+1}$ is the interarrival time between the nth and $(n+1)$th arriving customer, while $\boldsymbol{\tau}_n$ is the service time of the nth arriving customer. It is further assumed that

$$A(0+) = 0 \quad \text{and} \quad B(0+) = 0 .$$

The Laplace–Stieltjes transforms of $A(t)$ and $B(t)$ are defined by

$$\alpha(\rho) = \int_0^\infty e^{-\rho t} dA(t) , \quad \beta(\rho) = \int_0^\infty e^{-\rho t} dB(t) , \qquad \text{Re}\, \rho \geqq 0 .$$

Denote by $\boldsymbol{w}_n$ the total amount of work still to be handled by the server at the moment of arrival of the nth customer. Let K be a positive constant.

In the first model to be studied (model I) the nth arriving customer is admitted to the system if and only if

$$\boldsymbol{w}_n < K . \tag{1.1}$$

If the customer is admitted he waits for service, and service is in order of arrival. If the customer is refused, however, it is assumed that he never returns. Assuming that $\boldsymbol{w}_1 = 0$, the model I is described by the set of recurrence relations: for $n=1, 2, \ldots$,

$$\begin{aligned} \boldsymbol{w}_{n+1} &= [\boldsymbol{w}_n + \boldsymbol{\tau}_n - \boldsymbol{\sigma}_{n+1}]^+ \quad \text{if} \quad \boldsymbol{w}_n < K , \\ &= [\boldsymbol{w}_n - \boldsymbol{\sigma}'_{n+1}]^+ \quad \text{if} \quad \boldsymbol{w}_n \geqq K , \\ \boldsymbol{w}_1 &= 0 . \end{aligned} \tag{1.2}$$

The second model to be studied (model II) differs from model I in so far that the condition (1.1) is replaced by

$$w_n + \tau_n < K . \tag{1.3}$$

Again assuming that $w_1 = 0$ this model is described by the relations: for $n = 1, 2, \dots$,

$$\begin{aligned} w_{n+1} &= [w_n + \tau_n - \sigma_{n+1}]^+ \quad \text{if} \quad w_n + \tau_n < K , \\ &= [w_n - \sigma_{n+1}]^+ \quad\quad \text{if} \quad w_n + \tau_n \geqq K , \\ w_1 &= 0 . \end{aligned} \tag{1.4}$$

Other models with restricted accessibility are obtained if the constant K is replaced by a stochastic variable.

Let $\{u_n, n = 1, 2, \dots\}$ denote a sequence of independent, nonnegative and identically distributed variables, and assume that this family of variables is independent of the families $\{\sigma_n, n = 1, 2, \dots\}$ and $\{\tau_n = 1, 2, \dots\}$.

Model III is obtained from model I by replacing condition (1.1) by

$$w_n < u_n , \tag{1.5}$$

so that for this model: for $n = 1, 2, \dots$,

$$\begin{aligned} w_{n+1} &= [w_n + \tau_n - \sigma_{n+1}]^+ \quad \text{if} \quad w_n < u_n , \\ &= [w_n - \sigma_{n+1}]^+ \quad\quad \text{if} \quad w_n \geqq u_n , \\ w_1 &= 0 . \end{aligned} \tag{1.6}$$

For model IV the condition (1.3) is replaced by

$$w_n + \tau_n < u_n , \tag{1.7}$$

so that for this model: for $n = 1, 2, \dots$,

$$\begin{aligned} w_{n+1} &= [w_n + \tau_n - \sigma_{n+1}]^+ \quad \text{if} \quad w_n + \tau_n < u_n , \\ &= [w_n - \sigma_{n+1}]^+ \quad\quad \text{if} \quad w_n + \tau_n \geqq u_n , \\ w_1 &= 0 . \end{aligned}$$

In section 2 we shall study model I for $\alpha(\rho)$ and $\beta(\rho)$ both rational functions of ρ. The time dependent solution and stationary distribution are discussed. In section 3 we shall discuss model I for M/G/1 and particularize for M/M/1 and M/D/1. The case G/M/1 may be discussed along the same lines. Model II is first discussed for $\alpha(\rho)$ and $\beta(\rho)$ rational functions in section 4, particular results are given for M/M/1, while for M/D/1 such results are presented in section 5. Finally, model III is discussed in section 6, but only for M/M/1. Model IV is not discussed in this paper.

2. Model I

For model I we have for $n = 1, 2, \dots$, (cf. (1.2)),

$$\begin{aligned} w_{n+1} &= [w_n + \tau_n - \sigma_{n+1}]^+ \quad \text{if} \quad w_n < K , \\ &= [w_n - \sigma_{n+1}]^+ \quad\quad \text{if} \quad w_n \geqq K , \\ w_1 &= 0 . \end{aligned} \tag{2.1}$$

Hence for Re $\rho \geqq 0$,

$$\begin{aligned} E\{\exp(-\rho w_{n+1}) | w_1 = 0\} &= E\{\exp(-\rho [w_n + \tau_n - \sigma_{n+1}]^+)(w_n < K) | w_1 = 0\} \\ &\quad + E\{\exp(-\rho [w_n - \sigma_{n+1}]^+)(w_n \geqq K) | w_1 = 0\} , \end{aligned} \tag{2.2}$$

where for an event A we denote by (A) the indicator function of that event, so that

$$(w_n < K) = 1 \quad \text{if} \quad w_n < K\,, \qquad = 0 \quad \text{if} \quad w_n \geqq K\,.$$

Since by definition:

$$[x]^+ = 0 \text{ if } x \leqq 0, \quad = x \text{ if } x > 0\,,$$

we have

$$e^{-\rho[x]^+} = \frac{1}{2\pi i}\int_{C_\eta}\left(\frac{1}{\rho-\eta}+\frac{1}{\eta}\right)e^{-\eta x}d\eta, \quad \operatorname{Re}\rho > \operatorname{Re}\eta > 0\,, \tag{2.3}$$

where we use the notation

$$\int_{C_\eta} \dots\, d\eta = \lim_{\delta\to\infty}\int_{\varepsilon-i\delta}^{\varepsilon+i\delta} \dots\, d\eta \quad \text{with} \quad \varepsilon = \operatorname{Re}\eta\,.$$

Assuming that $\alpha(\rho)$ exists for $\operatorname{Re}\rho = 0-$, and noting that w_n, τ_n and σ_{n+1} are independent, it follows from (2.2) and (2.3) for $\operatorname{Re}\rho > \operatorname{Re}\xi = 0+$,

$$\begin{aligned}
E\{\exp(-\rho w_{n+1})|w_1=0\} &= \frac{1}{2\pi i}\int_{C_\xi}\left(\frac{1}{\rho-\xi}+\frac{1}{\xi}\right) \\
&\quad E\{\exp[-\xi(w_n+\tau_n-\sigma_{n+1})](w_n<K)|w_1=0\}\,d\xi \\
&\quad + \frac{1}{2\pi i}\int_{C_\xi}\left(\frac{1}{\rho-\xi}+\frac{1}{\xi}\right) \\
&\quad E\{\exp[-\xi(w_n-\sigma_{n+1})](w_n\geqq K)|w_1=0\}\,d\xi \\
&= \frac{1}{2\pi i}\int_{C_\xi}\left(\frac{1}{\rho-\xi}+\frac{1}{\xi}\right)\alpha(-\xi)\beta(\xi) \\
&\quad E\{\exp(-\xi w_n)(w_n<K)|w_1=0\}\,d\xi \\
&\quad + \frac{1}{2\pi i}\int_{C_\xi}\left(\frac{1}{\rho-\xi}+\frac{1}{\xi}\right)\alpha(-\xi) \\
&\quad E\{\exp(-\xi w_n)(w_n\geqq K)|w_1=0\}\,d\xi\,,
\end{aligned} \tag{2.4}$$

the reversal of integration and expectation is easily justified. Define for $|r| < 1$,

$$\Phi_1(r,\rho) = \sum_{n=1}^{\infty} r^n E\{\exp(-\rho w_n)(w_n<K)|w_1=0\}\,,$$

$$\Phi_2(r,\rho) = \sum_{n=1}^{\infty} r^n E\{\exp(-\rho w_n)(w_n\geqq K)|w_1=0\}\,, \quad \operatorname{Re}\rho \geqq 0\,,$$

$$\Phi\ (r,\rho) = \Phi_1(r,\rho)+\Phi_2(r,\rho) = \sum_{n=1}^{\infty} r^n E\{\exp(-\rho w_n)|w_1=0\}\,, \quad \operatorname{Re}\rho\geqq 0\,. \tag{2.5}$$

It follows from (2.4) for $|r|<1$, $\operatorname{Re}\rho > \operatorname{Re}\xi = 0+$,

$$\begin{aligned}
\Phi_1(r,\rho)+\Phi_2(r,\rho) &= \frac{r}{2\pi i}\int_{C_\xi}\left(\frac{1}{\rho-\xi}+\frac{1}{\xi}\right)\beta(\xi)\alpha(-\xi)\,\Phi_1(r,\xi)\,d\xi \\
&\quad + \frac{r}{2\pi i}\int_{C_\xi}\left(\frac{1}{\rho-\xi}+\frac{1}{\xi}\right)\alpha(-\xi)\,\Phi_2(r,\xi)\,d\xi + r\,.
\end{aligned} \tag{2.6}$$

Since for $|r|<1$, $\Phi_1(r,\rho)$ is an entire function of ρ, $\Phi_2(r,\rho)$ is analytic for $\operatorname{Re}\rho\geqq 0$ and

$$\lim_{|\rho|\to\infty} \Phi_1(r,\rho) = \sum_{n=1}^{\infty} r^n \Pr\{w_n=0\,|\,w_1=0\}\,, \quad -\tfrac{1}{2}\pi<\arg\rho<\tfrac{1}{2}\pi\,,$$
$$\lim_{|\rho|\to\infty} \Phi_2(r,\rho) = 0\,, \quad -\tfrac{1}{2}\pi<\arg\rho<\tfrac{1}{2}\pi\,, \tag{2.7}$$

it follows that for $|r|<1$, $\operatorname{Re}\rho>\operatorname{Re}\xi=0+$,

$$r = \frac{1}{2\pi i}\int_{C_\xi}\left(\frac{1}{\rho-\xi}+\frac{1}{\xi}\right)\{1-r\alpha(-\xi)\beta(\xi)\}\,\Phi_1(r,\xi)\,d\xi$$
$$+\frac{1}{2\pi i}\int_{C_\xi}\left(\frac{1}{\rho-\xi}+\frac{1}{\xi}\right)\{1-r\alpha(-\xi)\}\,\Phi_2(r,\xi)\,d\xi\,. \tag{2.8}$$

To obtain the solution of (2.8) it will be assumed for the present that $\alpha(\xi)$ and $\beta(\xi)$ are rational functions, i.e.

$$\alpha(\rho)=\frac{\alpha_1(\rho)}{\alpha_2(\rho)}\,,\quad \beta(\rho)=\frac{\beta_1(\rho)}{\beta_2(\rho)}\,, \tag{2.9}$$

where $\alpha_1(\rho)$, $\alpha_2(\rho)$, $\beta_1(\rho)$ and $\beta_2(\rho)$ are polynomials in ρ. Let m denote the degree of $\alpha_2(\rho)$, and n the degree of $\beta_1(\rho)$, then $\alpha_1(\rho)$ and $\beta_1(\rho)$ have a degree at most equal to $m-1$ and to $n-1$, respectively, since $A(0+)=B(0+)=0$.

From Rouché's theorem applied to the function of ρ,

$$1-r\alpha(-\rho)\beta(\rho) \quad \text{with} \quad |r|<1\,, \tag{2.10}$$

it follows easily that this function has exactly $m+n$ zero's $\delta_j(r)$, $j=1,\dots,m$; $\varepsilon_i(r)$, $i=1,\dots,n$. These zero's are continuous functions of r for $|r|\leqq 1$, and for $|r|<1$,

$$\operatorname{Re}\varepsilon_i(r)<0,\quad i=1,\dots,n\,;\qquad \operatorname{Re}\delta_j(r)>0,\quad j=1,\dots,m\,. \tag{2.11}$$

Define for $|r|\leqq 1$,

$$v_1(r)=\min_{1\leqq j\leqq m}\operatorname{Re}\delta_j(r)\,,\quad v_2(r)=\max_{1\leqq i\leqq n}\operatorname{Re}\varepsilon_i(r)\,. \tag{2.12}$$

For $|r|<1$, $f(r,\rho)$ and $g(r,\rho)$ are two polynomials in ρ of degree $m-1$ and $n-1$, respectively. These polynomials are uniquely determined by the condition that for $|r|<1$, the $m+n$ zero's of (2.10) are also zero's (and the only zero's) of

$$\frac{\rho f(r,\rho)}{\alpha_2(-\rho)}+\{\alpha_2(-\rho)-r\alpha_1(-\rho)\}\frac{g(r,\rho)}{\beta_2(\rho)}\mathrm{e}^{-\rho K}+r\,,\qquad |r|<1\,. \tag{2.13}$$

Defining for $|r|<1$,

$$H_-(r,\rho)=\frac{1-r\alpha(-\rho)\beta(\rho)}{\prod_{i=1}^{n}(\rho-\varepsilon_i(r))}\,\beta_2(\rho)\,,$$

$$H_+(r,\rho)=\frac{1-r\alpha(-\rho)\beta(\rho)}{\prod_{j=1}^{m}(\rho-\delta_j(r))}\,\alpha_2(-\rho)\,,$$

we shall show that the polynomials $f(r,\rho)$ and $g(r,\rho)$ are also determined by the following integral relations. For $v_1(r)>\operatorname{Re}\xi>0$, $\operatorname{Re}\xi>\operatorname{Re}\rho$, $|r|<1$,

$$f(r,\rho)=-\frac{\alpha_2(-\rho)H_-(r,\rho)}{2\pi i}\int_{C_\xi}\left\{(\alpha_2(-\xi)-r\alpha_1(-\xi))\frac{g(r,\xi)}{\beta_2(\xi)}\frac{\mathrm{e}^{-\xi K}}{\xi}+\frac{r}{\xi}\right\}\frac{1}{H_-(r,\xi)}\frac{d\xi}{\xi-\rho}\,, \tag{2.14}$$

and for $v_2(r)<\operatorname{Re}\eta<0$, $\operatorname{Re}\eta<\operatorname{Re}\rho$, $|r|<1$,

$$g(r,\rho)=\frac{\beta_2(\rho)H_+(r,\rho)}{2\pi i}\int_{C_\eta}\left\{\frac{\eta f(r,\eta)}{\alpha_2(-\rho)}+r\right\}e^{\eta K}\frac{1}{H_+(r,\eta)}\frac{d\eta}{\eta-\rho}\frac{1}{\alpha_2(-\eta)-r\alpha_1(-\eta)}. \tag{2.15}$$

For $g(r,\rho)$ a polynomial in ρ it is easily seen from (2.14) that $f(r,\rho)$ is a polynomial in ρ of degree $m-1$, and similarly from (2.15) it follows that $g(r,\rho)$ is a polynomial of degree $n-1$ if $f(r,\rho)$ is a polynomial. The integral in (2.14) is equal to the sum of its residues at its poles $\xi=\delta_j(r)$, $j=1,\ldots$, apart from a factor -1. Evaluation of this integral in (2.14) and taking $\rho=\delta_j(r)$, $j=1,\ldots,m$ leads to the condition (2.13) for $\rho=\delta_j(r), j=1,\ldots,m$. In the same way (2.15) leads to the condition (2.13) for $\rho=\varepsilon_i(r), i=1,\ldots,n$. Note that $\alpha_2(-\eta)-r\alpha_1(-\eta)$ has for $|r|<1$ no zero's with $\mathrm{Re}\,\eta<0$.

Next we shall prove that for $|r|<1$,

$$\Phi_1(r,\rho)=\frac{1}{1-r\alpha(-\rho)\beta(\rho)}\left\{\frac{\rho f(r,\rho)}{\alpha_2(-\rho)}+(\alpha_2(-\rho)-r\alpha_1(-\rho))\frac{g(r,\rho)}{\beta_2(\rho)}e^{-\rho K}+r\right\},$$

$$\Phi_2(r,\rho)=-\alpha_2(-\rho)\frac{g(r,\rho)}{\beta_2(\rho)}e^{-\rho K},\quad \mathrm{Re}\,\rho<0, \tag{2.16}$$

is a solution of (2.8). Substitution of (2.16) into (2.8) and noting that $\beta_2(\rho)$ and $\alpha_2(-\rho)$ have no zero's for $\mathrm{Re}\,\rho\geqq 0$ and $\mathrm{Re}\,\rho\leqq 0$, respectively, yields for $|r|<1$, $\mathrm{Re}\,\rho>\xi=0+$,

$$\frac{1}{2\pi i}\int_{C_\xi}\left(\frac{1}{\rho-\xi}+\frac{1}{\xi}\right)\{1-r\alpha(-\xi)\beta(\xi)\}\,\Phi_1(r,\xi)d\xi=(\alpha_2(-\rho)-r\alpha_1(-\rho))\frac{g(r,\rho)}{\beta_2(\rho)}e^{-\rho K}+r,$$

$$\frac{1}{2\pi i}\int_{C_\xi}\left(\frac{1}{\rho-\xi}+\frac{1}{\xi}\right)\{1-r\alpha(-\xi)\}\,\Phi_2(r,\xi)d\xi=-(\alpha_2(-\rho)-r\alpha_1(-\rho))\frac{g(r,\rho)}{\beta_2(\rho)}e^{-\rho K}.$$

Hence the expressions (2.16) satisfy (2.8).

It remains to prove that $\Phi_1(r,\rho)$ and $\Phi_2(r,\rho)$ as given by (2.16) are indeed the generating functions defined by (2.5).

From (2.5) it follows that for $|r|<1$, $\Phi_2(r,\rho)$ should be analytic for $\mathrm{Re}\,\rho\geqq 0$, while $\Phi_1(r,\rho)$ is an entire function of ρ of exponential type and order one. $E\{\exp(-\rho w_n)|\,w_n<K\}$ is the Laplace–Stieltjes transform of a distribution of a variable which is bounded to the left by zero, and from the right by K, whereas $E\{\exp(-\rho w_n)|\,w_n\geqq K\}$ is the transform of a distribution of a variable which is bounded to the left by K. These latter properties lead via (2.5) to properties of $\Phi_1(r,\rho)$ and $\Phi_2(r,\rho)$. It is readily verified that $\Phi_1(r,\rho)$ and $\Phi_2(r,\rho)$ as given by (2.16) possess the properties they should have on behalf of (2.5).

It is not difficult to prove that $\Phi_1(r,\rho)$ and $\Phi_2(r,\rho)$ as given by (2.16) have series expansions in r which are convergent for $|r|<1$, and which satisfy the set of recurrence relations (2.4). The solution of the system (2.4) is unique, since $E\{\exp(-\rho w_n)|\,w_n=0\}$ determines $E\{\exp(-\rho w_n)(w_n<K)\}$ and $E\{\exp(-\rho w_n)(w_n\geqq K)\}$ uniquely. Consequently, (2.16) represents the generating function of the solution of the set of equations (2.4). (For more details on the proof cf. also Cohen [1], ch. III,4.)

3. Model I for M/G/1

In this section we take with $\alpha>0$,

$$A(t)=1-e^{-t/\alpha},\quad t>0,$$
$$\quad\;\;=0\qquad\qquad t<0,$$

so that

$$\alpha(\rho)=\frac{1}{1+\alpha\rho},\quad \mathrm{Re}\,\rho>-\frac{1}{\alpha}.$$

Hence, $m=1$ and $f(r,\rho)$ is independent of ρ. We define

$$f(r)=f(r,\rho),\qquad |r|<1.$$

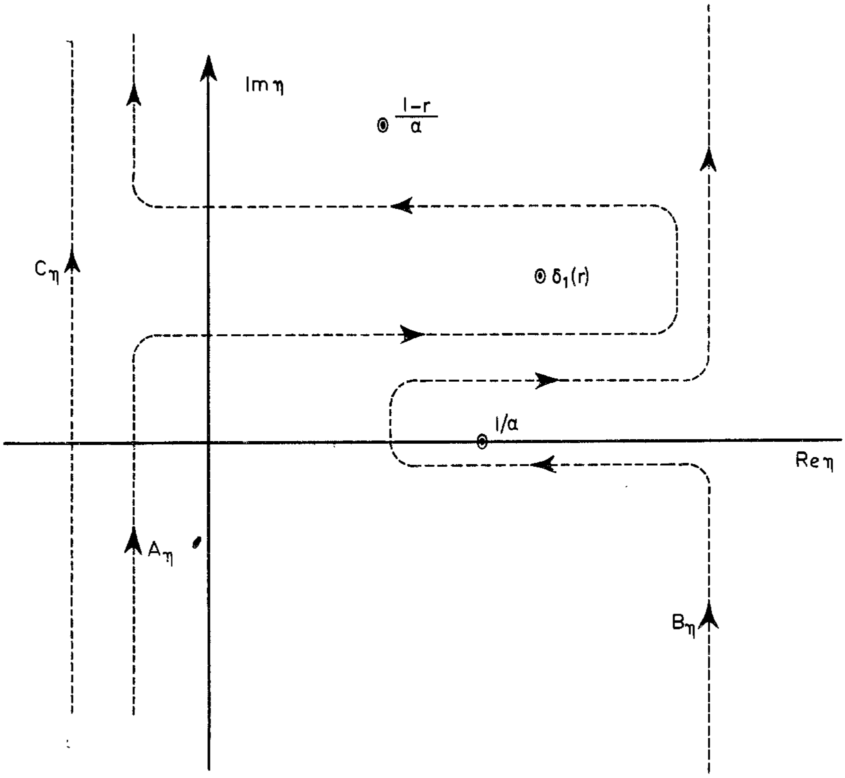
From (2.13) we have for $|r|<1$,

$$\frac{\delta_1(r)f(r)}{1-\alpha\,\delta_1(r)}+(1-r-\alpha\delta_1(r))\frac{g(r,\delta_1(r))}{\beta_2(\delta_1(r))}\exp(-\delta_1(r)K)+r=0\,. \tag{3.1}$$

From (2.15) it follows for $v_2(r)<\text{Re}\,\eta<0$, $\text{Re}\,\eta<\text{Re}\,\rho$, $|r|<1$,

$$\frac{g(r,\rho)}{\beta_2(\rho)}=\frac{1-\alpha\rho-r\beta(\rho)}{2\pi i\{\rho-\delta_1(r)\}}\int_{C_\eta}\left\{\frac{\eta f(r)}{1-\alpha\eta}+r\right\}\frac{\eta-\delta_1(r)}{1-\alpha\eta-r\beta(\eta)}\,\frac{e^{\eta K}}{1-r-\alpha\eta}\,\frac{d\eta}{\eta-\rho}\,. \tag{3.2}$$

In (3.2) we take $\rho=\delta_1(r)$ and replace the path C_η by A_η (see figure). Of the poles of the integrand



in (3.2) with $\rho=\delta_1(r)$ which are at the righthand side of C_η only the pole $\eta=\delta_1(r)$ is at the left-hand side of A_η.

It follows from (3.2) and (3.1) that

$$\frac{1}{2\pi i}\int_{A_\eta}\left\{\frac{\eta f(r)}{1-\alpha\eta}+r\right\}\frac{e^{\eta K}}{1-\alpha\eta-r\beta(\eta)}\,\frac{d\eta}{1-r-\alpha\eta}=0\,.$$

Hence

$$f(r)=-\frac{\dfrac{r}{2\pi i}\displaystyle\int_{A_\eta}\frac{e^{\eta K}}{1-\alpha\eta-r\beta(\eta)}\,\frac{d\eta}{1-r-\alpha\eta}}{\dfrac{1}{2\pi i}\displaystyle\int_{A_\eta}\frac{\eta}{1-\alpha\eta}\,\frac{e^{\eta K}}{1-\alpha\eta-r\beta(\eta)}\,\frac{d\eta}{1-r-\alpha\eta}}\,,\quad |r|<1\,. \tag{3.3}$$

Consequently, (3.2) and (3.3) determine $f(r)$ and $g(r,\rho)$, and hence (cf. (2.16)) the functions $\Phi_1(r,\rho)$ and $\Phi_2(r,\rho)$, i.e. for $\beta(\rho)$ a rational function of ρ the solution for model I has been obtained. Since in the derivation of the integral equation (2.8) no use has been made of the assumption that $\beta(\rho)$ is a rational function of ρ, and since it is readily verified that for general $\beta(\rho)$ the functions $\Phi_1(r,\rho)$ and $\Phi_2(r,\rho)$ determined by (3.2) and (2.16) also satisfy the integral equation (2.8), it follows easily that the solution obtained is valid for general service time distribution $B(t)$.

In (3.3) we replace the path A_η by the path B_η (see fig.). On B_η we have $\mathrm{Re}\,\eta > 0$, and the point $\eta = 1/\alpha$ is at the righthand side of this path, while for all $|r| < 1$ the points $(1-r)/\alpha$ and $\delta_1(r)$ are always at its lefthand side.

It follows for $|r| < 1$,

$$f(r) = \frac{-\alpha}{1-r} \frac{\dfrac{1-r}{1-\beta\{(1-r)/\alpha\}} \dfrac{e^{(1-r)K/\alpha}}{\alpha} + \dfrac{(1-r)r}{2\pi i} \displaystyle\int_{B_\eta} \frac{e^{\eta K}}{1-\alpha\eta - r\beta(\eta)} \frac{d\eta}{1-r-\alpha\eta}}{\dfrac{1-r}{1-\beta\{(1-r)/\alpha\}} \dfrac{e^{(1-r)K/\alpha}}{\alpha r^2} + \dfrac{\alpha}{2\pi i} \displaystyle\int_{B_\eta} \frac{(1-\alpha\eta)^{-1}\eta}{1-\alpha\eta - r\beta(\eta)} \frac{e^{\eta K}\,d\eta}{1-r-\alpha\eta}}, \quad |r| < 1 . \tag{3.4}$$

$f(r)$ is analytic for $|r| < 1$. From (3.4) it is not difficult to prove that $(1-r)f(r)$ has a series expansion in r which is convergent for $|r| < 1$. From (2.16) it follows for ρ real, $|r| < 1$,

$$\lim_{\rho\to\infty} \Phi_1(r, \rho) = -\frac{f(r)}{\alpha} = \sum_{n=1}^{\infty} r^n \Pr\{\mathbf{w}_n = 0 \mid \mathbf{w}_1 = 0\} .$$

By using the Abelian theorem for generating functions we obtain

$$\lim_{n\to\infty} \Pr\{\mathbf{w}_n = 0 \mid \mathbf{w}_1 = 0\} = \lim_{r\uparrow 1} (1-r)\frac{f(r)}{-\alpha} = \frac{1}{1 - \dfrac{a}{2\pi i} \displaystyle\int_{C_\eta} \frac{e^{\eta K}\alpha\, d\eta}{(1-\alpha\eta)\{1-\alpha\eta-\beta(\eta)\}}}, \quad a = \beta/\alpha , \tag{3.5}$$

for $\alpha^{-1} > \mathrm{Re}\,\eta > \delta_1(1)$. It is easily proved that for $r\uparrow 1$, $\lim \delta_1(r) = \delta_1(1) < \alpha^{-1}$.

Denote by $\mathbf{n}$ the number of customers which have arrived during a busy period, then

$$-\frac{f(r)}{\alpha} = \frac{E\{r^{\mathbf{n}}\}}{1-E\{r^{\mathbf{n}}\}}, \quad |r| < 1 . \tag{3.6}$$

From (3.6) it is easily found that for $\alpha^{-1} > \mathrm{Re}\,\eta > \delta_1(1)$,

$$E\{\mathbf{n}\} = 1 - \frac{a}{2\pi i} \int_{C_\eta} \frac{e^{\eta K}\alpha\, d\eta}{(1-\alpha\eta)\{1-\alpha\eta-\beta(\eta)\}} . \tag{3.7}$$

From (3.2) and the results obtained above it is easily shown that $(1-r)g(r, \rho)/\beta_2(\rho)$ has a limit for $r\uparrow 1$. We define

$$f = \lim_{r\uparrow 1} (1-r)f(r) , \qquad g(\rho) = \lim_{r\uparrow 1} (1-r)g(r, \rho)/\beta_2(\rho) . \tag{3.8}$$

If $a < 1$ then $\delta_1(1) = 0$. Since no pole of the integrand in (3.2) crosses the path C_η for $r\uparrow 1$ it follows from (3.2) if $a < 1$,

$$g(\rho) = \frac{1-\alpha\rho-\beta(\rho)}{\rho} \frac{1}{2\pi i} \int_{C_\eta} \frac{f}{-\alpha} \frac{\eta}{1-\alpha\eta} \frac{e^{\eta K}}{1-\alpha\eta-\beta(\eta)} \frac{d\eta}{n-\rho} , \tag{3.9}$$

for $\mathrm{Re}\,\rho \geqq 0$, $\mathrm{Re}\,\eta = 0-$.

If $a \geqq 1$, we rewrite (3.2) as (cf. derivation of (3.3)),

$$\begin{aligned}
\frac{g(r, \rho)}{\beta_2(\rho)} &= \frac{1-\alpha\rho - r\beta(\rho)}{\rho - \delta_1(r)} \frac{1}{2\pi i} \int_{A_\eta} \left\{ \frac{\eta f(r)}{1-\alpha\eta} + r \right\} \frac{\eta-\rho+\rho-\delta_1(r)}{1-\alpha\eta-r\beta(\eta)} \frac{e^{\eta K}}{1-r-\alpha\eta} \frac{d\eta}{\eta-\rho} \\
&= \{1-\alpha\rho-r\beta(\rho)\} \frac{1}{2\pi i} \int_{A_\eta} \left\{ \frac{\eta f(r)}{1-\alpha\eta} + r \right\} \frac{1}{1-\alpha\eta-r\beta(\eta)} \frac{e^{\eta K}}{1-r-\alpha\eta} \frac{d\eta}{\eta-\rho} \\
&= \{1-\alpha\rho-r\beta(\rho)\} \left[\left\{ \frac{1-r}{\alpha r} f(r) + r \right\} \frac{e^{(1-r)K/\alpha}}{r\{1-\beta((1-r)/\alpha)\}} \frac{1}{1-r-\alpha\rho} + \right. \\
&\qquad \left. + \frac{1}{2\pi i} \int_{B_\eta} \left\{ \frac{\eta f(r)}{1-\alpha\eta} + r \right\} \frac{1}{1-\alpha\eta-r\beta(\eta)} \frac{e^{\eta K}}{1-r-\alpha\eta} \frac{d\eta}{\eta-\rho} \right],
\end{aligned}$$

with $\mathrm{Re}\,\rho$ sufficiently large, i.e. on B_η we have $\mathrm{Re}\,\eta < \mathrm{Re}\,\rho$.

It follows that for $a \geqq 1$, $\alpha^{-1} > \text{Re}\,\eta > \delta_1(1)$, $\text{Re}\,\rho > \text{Re}\,\eta$,

$$g(\rho) = -\{1-\alpha\rho-\beta(\rho)\}\left[\frac{1}{\beta\rho}\left\{1+\frac{f}{\alpha}\right\} - \frac{1}{2\pi i}\int_{C_\eta}\frac{f}{-\alpha}\,\frac{1}{1-\alpha\eta}\,\frac{e^{\eta K}}{1-\alpha\eta-\beta(\eta)}\,\frac{d\eta}{\eta-\rho}\right]. \quad (3.10)$$

From the relations for $f(r)$ and $g(r,\rho)$ found above, the expressions for $\Phi_1(r,\rho)$, $\Phi_2(r,\rho)$ and $\Phi(r,\rho)$ may be obtained, and it is not difficult to prove that $E\{\exp(-\rho w_n)|w_1=0\}$, $E\{\exp(-\rho w_n)(w_n<K)|w_1=0\}$ and $E\{\exp(-\rho w_n)(w_n\geqq K)|w_1=0\}$ have limits for $n\to\infty$ (cf. the derivation of (3.5)).

It is found for $\text{Re}\,\rho \geqq 0$ that (cf. (2.16)),

$$\lim_{n\to\infty} E\{\exp(-\rho w_n)(w_n<K)|w_1=0\} = \frac{\alpha\rho}{1-\alpha\rho-\beta(\rho)}\left\{\frac{f}{\alpha} - (1-\alpha\rho)g(\rho)\exp(-\rho K)\right\},$$

$$\lim_{n\to\infty} E\{\exp(-\rho w_n)(w_n\geqq K)|w_1=0\} = -(1-\alpha\rho)g(\rho)\exp(-\rho K). \quad (3.11)$$

Since we have from (3.11) for $\rho=0$,

$$-\frac{1}{1-a}\left\{\frac{f}{\alpha} - g(0)\right\} - g(0) = 1,$$

it follows

$$g(0) = \frac{f}{\alpha a} + \frac{1-a}{a}. \quad (3.12)$$

An interesting quantity is the "congestion probability",

$$\lim_{n\to\infty} \Pr\{w_n \geqq K \mid w_1 = 0\}.$$

Obviously

$$\lim_{n\to\infty} \Pr\{w_n \geqq K \mid w_1 = 0\} = -g(0) = -a^{-1}\left\{\frac{f}{\alpha} + 1 - a\right\}$$

$$= \frac{1 + \dfrac{1-a}{2\pi i}\displaystyle\int_{C_\eta}\frac{e^{\eta K}}{1-\alpha\eta}\,\frac{\alpha\, d\eta}{1-\alpha\eta-\beta(\eta)}}{1 - \dfrac{a}{2\pi i}\displaystyle\int_{C_\eta}\frac{e^{\eta K}}{1-\alpha\eta}\,\frac{\alpha\, d\eta}{1-\alpha\eta-\beta(\eta)}}, \quad \delta_1(1) < \text{Re}\,\eta < \alpha^{-1}. \quad (3.13)$$

Next, we particularize some of the formulas derived above for negative exponentially distributed service times and for constant service time, respectively.

a) *Negative exponentially distributed service time, M/M/1.*
We now have with $\beta > 0$,

$$B(t) = 1 - e^{-t/\beta}, \quad t > 0,$$
$$= 0, \qquad t \leqq 0,$$

so that

$$\beta(\rho) = \frac{1}{1+\beta\rho}, \quad \text{Re}\,\rho > -1/\beta.$$

From (3.5) we obtain

$$\lim_{n\to\infty} \Pr\{w_n=0|w_1=0\} = \frac{1-a}{1-a^3 e^{-(1-a)K/\beta}}, \quad a \neq 1, \quad (3.14)$$

$$= \frac{1}{3+K/\beta}, \qquad a = 1,$$

and from (3.13)

$$\lim_{n\to\infty} \Pr\{w_n < K \mid w_1 = 0\} = \frac{1-a^2 e^{-(1-a)K/\beta}}{1-a^3 e^{-(1-a)K/\beta}}, \quad a \neq 1,$$

$$= \frac{2+K/\beta}{3+K/\beta}, \qquad a=1. \tag{3.15}$$

From (3.9) or (3.10) we obtain

$$g(\rho) = -\frac{a^2(1-a)}{1+\beta\rho} \frac{e^{-(1-a)K/\beta}}{1-a^3 e^{-(1-a)K/\beta}}. \tag{3.16}$$

From (3.14) and (3.16) we obtain (cf. (2.16)),

$$\Phi_1(\rho) = \lim_{r\uparrow 1} (1-r)\Phi_1(r, \rho)$$

$$= \frac{1-a}{1-a^3 e^{-(1-a)K/\beta}} \frac{1}{\beta\rho+1-a} \{1+\beta\rho - a^2(1-\alpha\rho) e^{-(\beta\rho+1-a)K/\beta}\}, \quad |\rho| < \infty,$$

$$\Phi_2(\rho) = \lim_{r\uparrow 1} (1-r)\Phi_2(r, \rho)$$

$$= \frac{(1-a)a^2}{1-a^3 e^{-(1-a)K/\beta}} \frac{1-\alpha\rho}{1+\beta\rho} e^{-(\beta\rho+1-a)K/\beta}, \quad \mathrm{Re}\,\rho \geqq 0,$$

$$\Phi(\rho) = \Phi_1(r) + \Phi_2(\rho)$$

$$= \frac{1-a}{1-a^3 e^{-(1-a)K/\beta}} \frac{1}{\beta\rho+1-a} \left\{1+\beta\rho - a^3 \frac{1-\alpha\rho}{1+\beta\rho} e^{-(\beta\rho+1-a)K/\beta}\right\}. \tag{3.17}$$

Hence

$$\lim_{n\to\infty} \Pr\{w_n < w \mid w_1 = 0\} = 0, \qquad w<0,$$

$$= \frac{1-a e^{-(1-a)w/\beta}}{1-a^3 e^{-(1-a)K/\beta}}, \qquad 0<w<K,$$

$$= \frac{1}{1-a^3 e^{-(1-a)K/\beta}} \{1-a^3 e^{-(1-a)K/\beta} - a(1-a)^2 e^{-(w-aK)/\beta}\}, \qquad w \geqq K. \tag{3.18}$$

b) *Constant service time, M/D/1.*
For this case

$$\beta(\rho) = e^{-\beta\rho}.$$

Since

$$\frac{e^{\eta K}}{(1-\alpha\eta)(1-\alpha\eta - e^{-\eta\beta})} = e^{\eta(K+\beta)} \left\{\frac{1}{1-\alpha\eta - e^{-\eta\beta}} - \frac{1}{1-\alpha\eta}\right\},$$

it follows from (3.5) and (3.13) for Re $\eta > \delta_1(1)$,

$$\lim_{n\to\infty} \Pr\{w_n = 0 \mid w_1 = 0\} = \frac{1}{1 - \frac{a}{2\pi i}\int_{C_\eta} e^{\eta(K+\beta)} \frac{\alpha\, d\eta}{1-\alpha\eta - e^{-\eta\beta}}},$$

$$\lim_{n\to\infty} \Pr\{w_n \geqq K \mid w_1 = 0\} = \frac{1 + \frac{1-a}{2\pi i}\int_{C_\eta} e^{\eta(K+\beta)} \frac{\alpha\, d\eta}{1-\alpha\eta - e^{-\eta\beta}}}{1 - \frac{a}{2\pi i}\int_{C_\eta} e^{\eta(K+\beta)} \frac{\alpha\, d\eta}{1-\alpha\eta - e^{-\eta\beta}}}. \tag{3.19}$$

For $a<1$ denote by w a stochastic variable with distribution the stationary waiting time distribution of the single server queue M/D/1. It is well known that for this case

$$E\{\exp(-\rho w)\}=(1-a)\frac{\alpha\rho}{e^{-\beta\rho}+\alpha\rho-1}, \quad \mathrm{Re}\,\rho\geqq 0. \tag{3.20}$$

Using the inversion formula for the Laplace–Stieltjes transform it follows that for Re $\eta>0$,

$$\begin{aligned}\Pr\{w<x\} &= \frac{1}{2\pi i}\int_{C_\eta}\frac{e^{\eta x}}{\eta}\frac{(1-a)\alpha\eta\,d\eta}{e^{-\beta\eta}+\alpha\eta-1}, && x>0,\\ &=0, && x<0.\end{aligned}$$

Hence, from (3.19) we have if $a<1$, so that $\delta_1(1)=0$,

$$\begin{aligned}\lim_{n\to\infty}\Pr\{w_n=0\,|\,w_1=0\} &= \frac{1-a}{1-a\Pr\{w\geqq K+\beta\}},\\ \lim_{n\to\infty}\Pr\{w_n\geqq K\,|\,w_1=0\} &= \frac{(1-a)\Pr\{w\geqq K+\beta\}}{1-a\ \Pr\{w\geqq K+\beta\}}.\end{aligned} \tag{3.21}$$

Obviously, the last relation represents the congestion probability.

4. Model II for M/M/1

For this model the mathematical formulation is given by

$$\begin{aligned}w_{n+1} &= [w_n+\tau_n-\sigma_{n+1}]^+ && \text{if}\quad w_n+\tau_n<K,\\ &= [w_n-\sigma_{n+1}]^+ && \text{if}\quad w_n+\tau_n\geqq K,\end{aligned} \tag{4.1}$$

for $n=1, 2, \dots$. We shall always take

$$w_1=0. \tag{4.2}$$

From (4.1) and (4.2) it follows that for $n=1, 2, \dots$,

$$w_n<K \text{ with probability one.} \tag{4.3}$$

It follows that

$$\begin{aligned}E\{\exp(-\rho w_{n+1})|w_1=0\} &= E\{\exp(-\rho[w_n+\tau_n+\sigma_{n+1}]^+)(w_n+\tau_n<K)|w_1=0\}\\ &\quad +E\{\exp(-\rho[w_n-\sigma_{n+1}]^+)(w_n+\tau_n\geqq K)|w_1=0\}\\ &= E\{\exp(-\rho[w_n-\sigma_{n+1}]^+)(w_n+\tau_n\geqq K)|w_1=0\}\\ &\quad +E\{(\exp(-\rho[w_n+\tau_n-\sigma_{n+1}]^+)\\ &\qquad -\exp(-\rho[w_n-\sigma_{n+1}]^+)(w_n+\tau_n<K)|w_1=0\}.\end{aligned} \tag{4.4}$$

In the sequel it will be assumed always that $\alpha(-\xi)$ exists for Re $\xi=0+$. From (4.4) we obtain for Re $\rho>$Re $\xi=0+$, Re $\eta=0+$,

$$\begin{aligned}E\{\exp(-\rho w_{n+1})|w_1=0\} &= \frac{1}{2\pi i}\int_{C_\xi}\left(\frac{1}{\rho-\xi}+\frac{1}{\xi}\right)\alpha(-\xi)E\{\exp(-\xi w_n)|w_1=0\}\,d\xi\\ &+\frac{1}{2\pi i}\int_{C_\xi}\left(\frac{1}{\rho-\xi}+\frac{1}{\xi}\right)\alpha(-\xi)E\{(\exp(-\xi(w_n+\tau_n))\\ &-\exp(-\xi w_n))(w_n+\tau_n<K)|w_1=0\}\mathrm{d}\xi\\ &=\frac{1}{2\pi i}\int_{C_\xi}\left(\frac{1}{\rho-\xi}+\frac{1}{\xi}\right)\alpha(-\xi)E\{\exp(-\xi w_n)|w_1=0\}\,d\xi+\end{aligned}$$

$$+\frac{1}{2\pi i}\int_{C_\eta}\left(\frac{1}{\rho-\xi}+\frac{1}{\xi}\right)\alpha(-\xi)\frac{d\xi}{2\pi i}\int_{C_\eta}E\left\{\frac{\exp(\eta(K-w_n-\tau_n))}{\eta}\right.$$
$$\left.\{\exp(-\xi(w_n+\tau_n))-\exp(-\xi w_n)\}\,|\,w_1=0\right\}d\eta$$
$$=\frac{1}{2\pi i}\int_{C_\xi}\left(\frac{1}{\rho-\xi}+\frac{1}{\xi}\right)\alpha(-\xi)E\{\exp(-\xi w_n)|w_1=0\}\,d\xi$$
$$+\frac{1}{2\pi i}\int_{C_\xi}\left(\frac{1}{\rho-\xi}+\frac{1}{\xi}\right)\alpha(-\xi)\frac{d\xi}{2\pi i}\int_{C_\eta}\frac{e^{\eta K}}{\eta}\{\beta(\xi+\eta)-\beta(\eta)\}$$
$$\times E\{\exp(-(\xi+\eta)w_n))|w_1=0\}\,d\eta\,. \quad (4.5)$$

In the derivation above we used the fact that w_n, τ_n and σ_{n+1} are independent; the reversal of integrations and expectation operator E is easily justified.

We may now introduce the generating function $\Sigma_{n=1}^{\infty} r^n E\{\exp(-\rho w_n)|w_1=0\}$ and derive from (4.5) an integral equation for this function. For $\alpha(\rho)$ and $\beta(\rho)$ rational functions of ρ it seems possible to solve this integral equation, however, the solution is very intricate. We shall restrict the discussion, therefore, to the simplest case:

$$\alpha(\rho)=\frac{1}{1+\alpha\rho},\quad \beta(\rho)=\frac{1}{1+\beta\rho},\quad \text{Re}\,\rho\geqq 0\,. \quad (4.6)$$

For this case the time dependent solution is still intricate and we shall therefore discuss only the stationary solution of the problem. Using a method as described in Cohen [1], ch. III, 4 it may be shown that

$$\lim_{n\to\infty}\Pr\{w_n<w\,|\,w_1=0\}$$

exists and that it is a proper probability distribution (cf. also Afanas'eva and Martynov [2]).

Define

$$\Phi(\rho)=\lim_{n\to\infty}E\{\exp(-\rho w_n)|w_1=0\}\,. \quad (4.7)$$

Below we shall determine the function $\Phi(\rho)$. The method to be used may be applied also if $\alpha(\rho)$ and $\beta(\rho)$ are rational functions of ρ.

For the stationary solution of the problem the integral equation (4.5) applies with $E\{\exp(-\rho w_{n+1})|w_1=0\}$ replaced by $\Phi(\rho)$. Using (4.6) we obtain for $\text{Re}\,\rho>\text{Re}\,\xi>0$, $\text{Re}\,\xi<\alpha^{-1}$, $\text{Re}\,\eta=0+$,

$$\frac{1}{2\pi i}\int_{C_\xi}\left(\frac{1}{\rho-\xi}+\frac{1}{\xi}\right)\left[\left\{1-\frac{1}{1-\alpha\xi}\right\}\Phi(\xi)d\xi\right.$$
$$\left.+\frac{\beta\xi}{1-\alpha\xi}\frac{d\xi}{2\pi i}\int_{C_\eta}\frac{e^{\eta K}}{\eta}\frac{1}{(1+\beta\eta)(1+(\xi+\eta)\beta)}\Phi(\xi+\eta)d\eta\right]=0\,.$$

With

$$a=\beta/\alpha \text{ and } \beta=1,$$

we obtain for $\text{Re}\,\eta=0+$, $\text{Re}\,\rho>\text{Re}\,\xi>0$, $\text{Re}\,\xi<\alpha^{-1}$,

$$\frac{1}{2\pi i}\int_{C_\xi}\frac{\rho}{\rho-\xi}\frac{1}{1-\xi/a}\left[\Phi(\xi)-\frac{a}{2\pi i}\int_{C_\eta}\frac{e^{\eta K}}{\eta}\frac{1}{1+\eta}\frac{1}{1+\eta+\xi}\Phi(\xi+\eta)\,d\eta\right]d\xi=0\,. \quad (4.8)$$

From (4.3) it follows that $\Phi(\rho)$ is an entire function of ρ of exponential type and order one; moreover,

$$\lim_{|\rho|\to\infty}\Phi(\rho)e^{\rho K}=0,\qquad \tfrac{1}{2}\pi<\arg\rho<1\tfrac{1}{2}\pi,$$
$$\lim_{|\rho|\to\infty}\Phi(\rho)=\text{constant},\quad -\tfrac{1}{2}\pi<\arg\rho<\tfrac{1}{2}\pi\,. \quad (4.9)$$

Hence, by contour integration, for Re $\rho>$Re $\xi>0$, Re $\xi<a$,

$$\frac{1}{2\pi i}\int_{C_\xi}\frac{1}{\rho-\xi}\,\frac{1}{1-\xi/a}\left[\frac{1+\xi-a}{1+\xi}\,\Phi(\xi)-\frac{a\,\mathrm{e}^{-(1+\xi)K}}{\xi(\xi+1)}\,\Phi(-1)+\frac{a}{\xi}\,\mathrm{e}^{-K}\Phi(\xi-1)\right]d\xi=0\,.$$

Again by contour integration

$$\left\{\frac{1+\rho-a}{1+\rho}\,\Phi(\rho)-\frac{a\,\mathrm{e}^{-(1+\rho)K}}{\rho(\rho+1)}\,\Phi(-1)+\frac{a}{\rho}\,\mathrm{e}^{-K}\Phi(\rho-1)\right\}\frac{1}{1-\rho/a}$$
$$+\left\{\frac{1}{1+a}\,\Phi(a)-\frac{a\,\mathrm{e}^{-(1+a)K}}{a(1+a)}\,\Phi(-1)+\mathrm{e}^{-K}\Phi(a-1)\right\}\frac{a}{\rho-a}=0\,. \tag{4.10}$$

The relation (4.10) must be valid for all finite $|\rho|$ since $\Phi(\rho)$ is an entire function. Note that $\rho=0$ and $\rho=-1$ are no singular points of (4.10). Hence

$$\frac{1+\rho-a}{1-\rho}\,\Phi(\rho)+\frac{a}{\rho}\,\mathrm{e}^{-K}\Phi(\rho-1)-\frac{a\,\mathrm{e}^{-(1+\rho)K}}{\rho(\rho+1)}\,\Phi(-1)=C\,, \tag{4.11}$$

where C is a constant, i.e. independent of ρ. Since the coefficient of $\Phi(\rho)$ is zero for $\rho=a-1$, it is seen that the general solution of (4.11) will have a singularity at $\rho=a-1$. $\Phi(\rho)$ should be an entire function, however, i.e. the constant C should be chosen such that the solution of (4.11) has no singularity at $\rho=a-1$.

Defining

$$p(\rho)=\frac{a}{\rho}\,\frac{1+\rho}{a-\rho-1}\,\mathrm{e}^{-K},\qquad q(\rho)=\frac{a}{\rho}\,\frac{\mathrm{e}^{-(1+\rho)K}}{a-\rho-1},\qquad r(\rho)=\frac{1+\rho}{a-\rho-1},$$

we have

$$\Phi(\rho)=p(\rho)\Phi(\rho-1)-q(\rho)\Phi(-1)-r(\rho)C\,. \tag{4.12}$$

From (4.12) we obtain

$$\Phi(\rho)=-q(\rho)\Phi(-1)-r(\rho)C+p(\rho)[-q(\rho-1)\Phi(-1)-r(\rho-1)C+p(\rho-1)\Phi(\rho-2)]\,,$$

and for $n=0, 1, 2, \dots$,

$$\Phi(\rho)=-\Phi(-1)\left[q(\rho)+\sum_{i=1}^{n+1}q(\rho-i)\prod_{j=0}^{i-1}p(\rho-j)\right]$$
$$-C\left[r(\rho)+\sum_{i=1}^{n+1}r(\rho-i)\sum_{j=0}^{i-1}p(\rho-j)\right]+\prod_{j=0}^{n+1}p(\rho-j)\Phi(p-n-2)\,. \tag{4.13}$$

Since

$$\prod_{j=0}^{i-1}p(\rho-j)=a^i\mathrm{e}^{-iK}\,\frac{\rho+1}{\rho-i+1}\prod_{j=0}^{i-1}\frac{1}{a-1-\rho+j},$$

$$r(\rho-i)\prod_{j=0}^{i-1}p(\rho-j)=a^i\mathrm{e}^{-iK}(\rho+1)\prod_{j=0}^{i}\frac{1}{a-1-\rho+j},$$

$$q(\rho-i)\prod_{j=0}^{i-1}p(\rho-j)=a^{i+1}\mathrm{e}^{-(1+\rho)K}\left(\frac{1}{\rho-i}-\frac{1}{\rho-i+1}\right)(\rho+1)\prod_{j=0}^{i}\frac{1}{a-1-\rho+j}, \tag{4.14}$$

and since on behalf of (4.9),

$$\lim_{n\to\infty}\prod_{j=0}^{n+1}p(\rho-j)\Phi(\rho-n-2)=0$$

it follows from (4.13),

$$\Phi(\rho) = -\Phi(-1)\left[q(\rho) + \sum_{i=1}^{\infty} q(\rho-i)\prod_{j=0}^{i-1} p(\rho-j)\right]$$
$$-C\left[r(\rho) + \sum_{i=1}^{\infty} r(\rho-i)\prod_{j=0}^{i-1} p(\rho-j)\right]. \tag{4.15}$$

Inserting the relations (4.14) in (4.15) yields

$$\Phi(\rho) = \Phi(-1)\mathrm{e}^{-(1+\rho)K}\left\{1+(1+\rho)\sum_{i=0}^{\infty}\frac{a^i}{i!}\frac{\Gamma(a-\rho-1)\Gamma(i+1)}{\Gamma(a-\rho+i)}\right\}$$
$$-C(1+\rho)\sum_{i=0}^{\infty}\frac{a^i}{i!}\mathrm{e}^{-iK}\frac{\Gamma(a-\rho-1)\Gamma(i+1)}{\Gamma(a-\rho+i)}. \tag{4.16}$$

The constant C is determined by the condition that $\Phi(\rho)$ should be an entire function.

Taking in (4.16) $\rho=a-1$ the condition that $\Phi(\rho)$ is an entire function yields

$$C = \mathrm{e}^{a(1-K)-a\mathrm{e}^{-K}}\Phi(-1), \tag{4.17}$$

with C given by (4.17) it is readily verified that $\Phi(\rho)$ has no poles for finite $|\rho|$.

The norming condition requires

$$\Phi(0) = 1,$$

and by this condition $\Phi(-1)$ is determined. It follows

$$\Phi(-1) = \left[\left\{1+\sum_{i=0}^{\infty} a^i\frac{\Gamma(a-1)}{\Gamma(a+i)}\right\}\mathrm{e}^{-K} - \mathrm{e}^{a(1-K)-a\mathrm{e}^{-K}}\sum_{i=0}^{\infty} a^i\mathrm{e}^{-iK}\frac{\Gamma(a-1)}{\Gamma(a+i)}\right]^{-1}, \tag{4.18}$$

for $a\neq 1$. From (4.16) we obtain for the probability that an arriving customer has zero waiting time

$$\lim_{n\to\infty} \Pr\{w_n=0\,|\,w_1=0\} = C. \tag{4.19}$$

The probability of congestion is given by

$$\lim_{n\to\infty} \Pr\{w_n+\tau_n \geqq K\,|\,w_1=0\} = 1-\lim_{n\to\infty} \Pr\{w_n+\tau_n < K\,|\,w_1=0\}.$$

Using the inversion formula for the Laplace–Stieltjes transform we obtain for $\mathrm{Re}\,\rho>0$, by noting that $\Phi(\rho)$ is an entire function,

$$\lim_{n\to\infty} \Pr\{w_n+\tau_n \geqq K\,|\,w_1=0\} = 1 - \frac{1}{2\pi i}\int_{C_\rho}\frac{\mathrm{e}^{\rho K}}{\rho}\frac{\Phi(\rho)}{1+\rho}d\rho$$
$$= 1 - \frac{1}{2\pi i}\int_{C_\rho}\frac{\mathrm{e}^{\rho K}}{\rho(\rho+1)}\Phi(-1)\mathrm{e}^{-(1+\rho)K}d\rho$$
$$-\frac{1}{2\pi i}\int_{C_\rho}\frac{\mathrm{e}^{\rho K}}{\rho}d\rho\sum_{i=0}^{\infty}\frac{a^i}{(a-\rho-1)\dots(a-\rho+i-1)}\{\Phi(-1)\mathrm{e}^{-(1+\rho)K} - C\mathrm{e}^{-iK}\}$$
$$= \Phi(-1)\mathrm{e}^{-K}. \tag{4.20}$$

To obtain the stationary distribution $W(x)$ we start from

$$W(x)-1 = \frac{1}{2\pi i}\int_{C_\rho}\frac{\mathrm{e}^{\rho x}}{\rho}\Phi(\rho)\,d\rho, \qquad \mathrm{Re}\,\rho<0,\ \mathrm{Re}\,\rho<a-1.$$

It follows for $0<x<K$, $\mathrm{Re}\,\rho<a-1$, $\mathrm{Re}\,\rho<0$,

$$W(x)-1=\frac{1}{2\pi i}\int_{C_\rho}\frac{e^{\rho x}}{\rho}\,d\rho\left[\Phi(-1)e^{-(1+\rho)K}\right.$$

$$+(\rho+1)\Phi(-1)e^{-(1+\rho)K}\sum_{i=0}^{\infty}\frac{a^i}{(a-\rho-1)\dots(a-\rho+i-1)}$$

$$\left.-(\rho+1)C\sum_{i=0}^{\infty}\frac{a^i e^{-iK}}{(a-\rho-1)\dots(a-\rho+i-1)}\right]$$

$$=-\Phi(-1)e^{-K}-\Phi(-1)e^{-K}\sum_{i=0}^{\infty}\frac{a^i}{(a-1)a\dots(a+i-1)}$$

$$+\Phi(-1)\frac{a}{a-1}e^{(a-1)x-aK}\left\{1+\sum_{i=1}^{\infty}\frac{a^i}{i!}\right\}$$

$$+\Phi(-1)\frac{a+1}{a}e^{ax-(a+1)K}\left\{-a+\sum_{i=2}^{\infty}\frac{a^i}{-1\cdot 1\dots i-1}\right\}$$

$$+\Phi(-1)\frac{a+2}{a+1}e^{(a+1)x-(a+2)K}\left\{\frac{a^2}{-2\cdot-1}+\sum_{i=3}^{\infty}\frac{a^i}{-2\cdot-1\cdot 1\cdot 2\dots(i-2)}\right\}$$

$$+\Phi(-1)\frac{a+3}{a+2}e^{(a+2)x-(a+3)K}\left\{\frac{a^3}{-3\cdot-2\cdot-1}\right.$$

$$\left.+\sum_{i=4}^{\infty}\frac{a^i}{-3\cdot-2\cdot-1\cdot 1\cdot 2\dots(i-3)}\right\}$$

$$+\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots .$$

It follows

$$W(x)=1-\Phi(-1)e^{-K}\left\{1+\sum_{i=0}^{\infty}\frac{a^i\Gamma(a-1)}{\Gamma(a+i)}\right.$$

$$\left.-e^{a+(a-1)(x-K)}\left\{e^{-ae^{+(x-K)}}+\sum_{j=0}^{\infty}\frac{(-1)^j}{a+j-1}\frac{a^j}{j!}e^{+j(x-K)}\right\}\right\}. \tag{4.21}$$

Till now it has been assumed that $a\neq 1$. Next we shall consider the case that $a=1$. We rewrite the norming condition as follows

$$\{\Phi(-1)e^{-K}\}^{-1}=1+\frac{1}{a-1}\left\{2+\sum_{i=2}^{\infty}\frac{a^{i-1}}{(a-1)\dots(a+i-1)}\right\}$$

$$-\frac{e^{(1-a)K+a(1-e^{-K})}}{a-1}\left\{1+e^{-K}+\sum_{i=2}^{\infty}\frac{a^{i-1}e^{-iK}}{(a+1)\dots(a+i-1)}\right\}.$$

Letting $a\to 1$ and applying Hospital's rule yields

$$\{\Phi(-1)e^{-K}\}^{-1}=1+\sum_{i=2}^{\infty}\frac{i-1}{i!}-\sum_{i=2}^{\infty}\frac{1}{i!}\left\{\tfrac{1}{2}+\dots+\frac{1}{i}\right\}$$

$$-e^{(1-e^{-K})}\left\{\sum_{i=2}^{\infty}\frac{(i-1)e^{-iK}}{i!}-\sum_{i=2}^{\infty}\frac{e^{-iK}}{i!}\left\{\tfrac{1}{2}+\dots+\frac{1}{i}\right\}\right\}$$

$$+(K-1+e^{-K})e^{1-e^{-K}}\sum_{i=0}^{\infty}\frac{e^{-iK}}{i!}.$$

Hence

$$\{\Phi(-1)\mathrm{e}^{-K}\}^{-1}=1+\mathrm{e}-\sum_{i=1}^{\infty}\frac{1}{i!}\left(1+\ldots+\frac{1}{i}\right)$$
$$-\mathrm{e}^{(1-\mathrm{e}^{-K})}\left(\mathrm{e}^{-K+\mathrm{e}^{-K}}-\sum_{i=1}^{\infty}\frac{\mathrm{e}^{-iK}}{i!}\left\{1+\ldots+\frac{1}{i}\right\}\right)+(K-1+\mathrm{e}^{-K})\mathrm{e}\,.$$

Since (cf. Erdélyi, [3]),

$$Ei(-x)=-\int_x^{\infty}t^{-1}\mathrm{e}^{-t}dt=\gamma+\log x-\mathrm{e}^{-x}\sum_{i=1}^{\infty}\frac{x^i}{i!}\left(1+\ldots+\frac{1}{i}\right),$$

it follows

$$\begin{aligned}\{\Phi(-1)\mathrm{e}^{-K}\}^{-1}&=1+\mathrm{e}+\mathrm{e}Ei(-1)-\mathrm{e}\gamma-\mathrm{e}\{\mathrm{e}^{-K}+Ei(-\mathrm{e}^{-K})-\gamma+K\}+\mathrm{e}(K-1+\mathrm{e}^{-K})\\&=1+\mathrm{e}\{Ei(-1)-Ei(-\mathrm{e}^{-K})\}\,.\end{aligned}$$

Hence for $a=1$, note that we have taken here always $\beta=1$,

$$\Phi(-1)\mathrm{e}^{-K}=\frac{1}{1+\mathrm{e}\int_{\mathrm{e}^{-K}}^{1}t^{-1}\mathrm{e}^{-t}dt}, \tag{4.22}$$

$$C=\mathrm{e}^{(1-K)-\mathrm{e}^{-K}}\Phi(-1)\,.$$

It is of some interest to consider the probabilities $\lim_{n\to\infty}\Pr\{w_n=0\,|\,w_1=0\}$ and $\lim_{n\to\infty}\Pr\{w_n+\tau_n\geqq K\,|\,w_1=0\}$ for $K\to\infty$. From (4.19), (4.20) and (4.22) we obtain for $K\to\infty$

$$\begin{aligned}\Phi(-1)\mathrm{e}^{-K}&\to 0\,, & C&\to 1-a \quad \text{for}\quad a<1\,,\\ &\to 0\,, & &\to 0 \quad \text{for}\quad a=1\,,\\ &\to\frac{1}{1+\sum_{i=0}^{\infty}\frac{a^i}{(a-1)a\ldots(a+i-1)}}\,, & &\to 0 \quad \text{for}\quad a>1\,.\end{aligned} \tag{4.23}$$

The last relation of (4.23) is of particular interest since it shows clearly the influence of congestion on the behaviour of the queue with traffic intensity a exceeding one.

5. Model II for M/D/1

For the present case the service time is constant and equal to β, while

$$\beta(\rho)=\mathrm{e}^{-\beta\rho}\,.$$

Obviously, the case $\beta>K$ is of no interest, since no customer will be admitted to the system. Henceforth it will be assumed that $K\geqq\beta$.

Define

$$H=K-\beta\,.$$

It then follows from (1.4) that

$$\begin{aligned}w_{n+1}&=[w_n+\beta-\sigma_{n+1}]^+ \quad \text{if}\quad w_n<K-\beta=H\,,\\&=[w_n-\sigma_{n+1}]^+ \qquad\;\; \text{if}\quad w_n\geqq K-\beta=H\,.\end{aligned}$$

$$w_1=0\,.$$

These relations, however, are identical with (1.2) if K is replaced by H. Hence for the present case the results of section 2 and 3 can be used.

We shall here consider only the probability that the system is empty at a moment of arrival and the probability of congestion, both for the stationary situation.

Noting that

$$\frac{1}{1-\alpha\eta-e^{-\beta\eta}}\,\frac{e^{-\eta\beta}}{1-\alpha\eta}=\frac{1}{1-\alpha\eta-e^{-\eta\beta}}-\frac{1}{1-\alpha\eta}$$

it follows from (3.5) and (3.13) with K replaced by $K-\beta$ that for $\mathrm{Re}\,\eta>\delta_1(1)$,

$$\lim_{n\to\infty}\Pr\{w_n=0\,|\,w_1=0\}=\frac{1}{1-\dfrac{a}{2\pi i}\displaystyle\int_{C_\eta}\frac{e^{\eta K}\alpha\,d\eta}{1-\alpha\eta-e^{-\beta\eta}}}, \tag{5.1}$$

$$\lim_{n\to\infty}\Pr\{w_n+\beta\geqq K\,|\,w_1=0\}=\frac{1+\dfrac{1-a}{2\pi i}\displaystyle\int_{C_\eta}e^{\eta K}\frac{\alpha\,d\eta}{1-\alpha\eta-e^{-\beta\eta}}}{1-\dfrac{a}{2\pi i}\displaystyle\int_{C_\eta}e^{\eta K}\frac{\alpha\,d\eta}{1-\alpha\eta-e^{-\beta\eta}}}. \tag{5.2}$$

Suppose for the present that $a<1$ and denote by w a stochastic variable with distribution the stationary waiting time distribution of the M/D/1 system.

It is well known that

$$E\{\exp(-\rho w)\}=(1-a)\frac{\alpha\rho}{e^{-\rho\beta}+\alpha\rho-1},\qquad \mathrm{Re}\,\rho\geqq 0\,. \tag{5.3}$$

Using the inversion formula for the Laplace–Stieltjes transform we have

$$\Pr\{w<x\}=\frac{1}{2\pi i}\int_{C_\eta}\frac{e^{\eta x}}{\eta}\,\frac{(1-a)\alpha\eta\,d\eta}{e^{-\beta\eta}+\alpha\eta-1},\qquad \mathrm{Re}\,\eta>0\,.$$

Consequently, we may rewrite (5.1) and (5.2) as

$$\lim_{n\to\infty}\Pr\{w_n=0\,|\,w_1=0\}=\frac{1-a}{1-a\Pr\{w\geqq K\}},$$

$$\lim_{n\to\infty}\Pr\{w_n+\beta\geqq K\,|\,w_1=0\}=\frac{(1-a)\Pr\{w\geqq K\}}{1-a\Pr\{w\geqq K\}}. \tag{5.4}$$

If $a>1$ then $\delta_1(1)>0$ so that the integrands in (5.1) and (5.2) have a pole in the righthalfplane. Using this fact it is easily derived from (5.1), (5.2) that for $K\to\infty$,

$$\begin{aligned}\lim_{n\to\infty}\Pr\{w_n=0\,|\,w_1=0\}&\to 1-a &&\text{if}\quad a<1\,,\\ &\to 0 &&\text{if}\quad a\geqq 1\,,\end{aligned} \tag{5.5}$$

$$\begin{aligned}\lim_{n\to\infty}\Pr\{w_n+\beta\geqq K\,|\,w_1=0\}&\to 0 &&\text{if}\quad a\leqq 1\,,\\ &\to 1-\frac{1}{a} &&\text{if}\quad a>1\,.\end{aligned} \tag{5.6}$$

6. Model III

For model III we have for $n=1,2,\ldots$, (cf. 1.6),

$$\begin{aligned}w_{n+1}&=[w_n+\tau_n-\sigma_{n+1}]^+ &&\text{if}\quad w_n<u_n\,,\\ &=[w_n-\sigma_{n+1}]^+ &&\text{if}\quad w_n\geqq u_n\,,\\ w_1&=0\,.\end{aligned} \tag{6.1}$$

For the distribution of $\boldsymbol{u}_n$ we shall take here for all $n=1, 2, \dots$,

$$\Pr\{\boldsymbol{u}_n < t\} = 1-e^{-t/c}, \quad t>0,\ c>0, \qquad =0, \quad t<0.$$

From (6.1) we obtain for Re $\rho \geqq 0$, $n=1, 2, \dots$,

$$\begin{aligned} E\{\exp(-\rho \boldsymbol{w}_{n+1})|\boldsymbol{w}_1=0\} &= E\{\exp(-\rho[\boldsymbol{w}_n+\boldsymbol{\tau}_n-\boldsymbol{\sigma}_{n+1}]^+)(\boldsymbol{w}_n<\boldsymbol{u}_n)|\boldsymbol{w}_1=0\} \\ &\quad + E\{\exp(-\rho[\boldsymbol{w}_n-\boldsymbol{\sigma}_{n+1}]^+)(\boldsymbol{w}_n \geqq \boldsymbol{u}_n)|\boldsymbol{w}_1=0\} \\ &= E\{[\exp(-\rho[\boldsymbol{w}_n+\boldsymbol{\tau}_n-\boldsymbol{\sigma}_{n+1}]^+)-\exp(-\rho[\boldsymbol{w}_n-\boldsymbol{\sigma}_{n+1}]^+)] (\boldsymbol{w}_n<\boldsymbol{u}_n)|\boldsymbol{w}_1=0\} \\ &\quad + E\{\exp(-\rho[\boldsymbol{w}_n-\boldsymbol{\sigma}_{n+1}]^+)|\boldsymbol{w}_1=0\}. \end{aligned} \tag{6.2}$$

We have since $\boldsymbol{w}_n$ and $\boldsymbol{u}_n$ are independent by assumption

$$\begin{aligned} E\{\exp(-\rho\boldsymbol{w}_n)(\boldsymbol{w}_n<\boldsymbol{u}_n)\} &= \int_0^\infty E\{\exp(-\rho\boldsymbol{w}_n)(\boldsymbol{w}_n<\boldsymbol{u}_n)\}\, d_u \Pr\{\boldsymbol{u}_n<u\} \\ &= \int_{u=0}^\infty \int_{w=0}^u \exp(-\rho w) d_w \Pr\{\boldsymbol{w}_n<w\}\, d_u \Pr\{\boldsymbol{u}_n<u\} \\ &= \int_{w=0}^\infty \int_{u=w}^\infty \exp(-\rho w) d_w \Pr\{\boldsymbol{w}_n<w\}\, d_u \Pr\{\boldsymbol{u}_n<u\} \\ &= \int_{w=0}^\infty \exp\left\{-\left(\rho+\frac{1}{c}\right)w\right\} d\Pr\{\boldsymbol{w}_n<w\} \\ &= E\left\{\exp\left\{-\left(\rho+\frac{1}{c}\right)\boldsymbol{w}_n\right\}\right\}. \end{aligned} \tag{6.3}$$

Assuming that $\alpha(0-)$ exists we obtain from (6.2) for Re $\rho >$ Re $\xi=0+$, $n=1, 2, \dots$,

$$\begin{aligned} E\{\exp(-\rho\boldsymbol{w}_{n+1})|\boldsymbol{w}_1=0\} &= \frac{1}{2\pi i}\int_{C_\xi}\left(\frac{1}{\rho-\xi}+\frac{1}{\xi}\right)E\{\exp(-\xi(\boldsymbol{w}_n-\boldsymbol{\sigma}_{n+1}))|\boldsymbol{w}_1=0\}d\xi \\ &\quad + \frac{1}{2\pi i}\int_{C_\xi}\left(\frac{1}{\rho-\xi}+\frac{1}{\xi}\right)E\{[\exp(-\xi(\boldsymbol{w}_n+\boldsymbol{\tau}_n-\boldsymbol{\sigma}_{n+1})) \\ &\quad -\exp(-\xi(\boldsymbol{w}_n-\boldsymbol{\sigma}_{n+1}))](\boldsymbol{w}_n<\boldsymbol{u}_n)|\boldsymbol{w}_1=0\}d\xi, \end{aligned} \tag{6.4}$$

the reversal of integration and expectation operator is easily justified. Since $\boldsymbol{w}_n$, $\boldsymbol{\tau}_n$, $\boldsymbol{\sigma}_{n+1}$ and $\boldsymbol{u}_n$ are independent variables, it follows from (6.3) and (6.4) for Re $\rho>$ Re $\xi=0+$, $n=1, 2, \dots$,

$$\begin{aligned} E\{\exp(-\rho\boldsymbol{w}_{n+1})|\boldsymbol{w}_1=0\} &= \frac{1}{2\pi i}\int_{C_\xi}\left(\frac{1}{\rho-\xi}+\frac{1}{\xi}\right)\alpha(-\xi)E\{\exp(-\xi\boldsymbol{w}_n)|\boldsymbol{w}_1=0\}d\xi \\ &\quad - \frac{1}{2\pi i}\int_{C_\xi}\left(\frac{1}{\rho-\xi}+\frac{1}{\xi}\right)\alpha(-\xi)\{1-\beta(\xi)\}E\left\{\exp\left(-\left(\xi+\frac{1}{c}\right)\boldsymbol{w}_n\right)|\boldsymbol{w}_1=0\right\}d\xi. \end{aligned} \tag{6.5}$$

Defining for $|r|<1$, Re $\rho \geqq 0$,

$$\Phi(r,\rho) \overset{\text{def}}{=} \sum_{n=1}^\infty r^n E\{\exp(-\rho\boldsymbol{w}_n)|\boldsymbol{w}_1=0\}, \tag{6.6}$$

we obtain from (6.5) for $|r|<1$, Re $\rho>$ Re $\xi=0+$,

$$\frac{1}{2\pi i}\int_{C_\xi}\left(\frac{1}{\rho-\xi}+\frac{1}{\xi}\right)\{1-r\alpha(-\xi)\}\Phi(r,\xi)d\xi + \frac{1}{2\pi i}\int_{C_\xi}\left(\frac{1}{\rho-\xi}+\frac{1}{\xi}\right)r\alpha(-\xi)(1-\beta(\xi))\Phi\left(r,\xi+\frac{1}{c}\right)d\xi = r. \tag{6.7}$$

From now on it will be assumed that the interarrival times have a negative exponential distribution so that

$$\alpha(\rho)=\frac{1}{1-\alpha\rho},\quad \operatorname{Re}\rho>-\frac{1}{\alpha},\qquad \alpha>0. \tag{6.8}$$

Consequently, for $|r|<1$, $\operatorname{Re}\rho>\operatorname{Re}\xi=0+$,

$$\frac{1}{2\pi i}\int_{C_\xi}\frac{\rho}{\rho-\xi}\frac{1}{1-\alpha\xi}\left[r\frac{1-\beta(\xi)}{\xi}\Phi\left(r,\xi+\frac{1}{c}\right)+\frac{1-r-\alpha\xi}{\xi}\Phi(r,\xi)-\frac{1-\alpha\xi}{\xi}r\right]d\xi=0. \tag{6.9}$$

Denote by $\Psi(r,\xi)$ the expression between square brackets in the integral of (6.9). Obviously, $\Psi(r,\xi)$ is an analytic function of ξ for $|r|<1$, $\operatorname{Re}\xi>0$. Consequently,

$$0=\frac{1}{2\pi i}\int_{C_\xi}\frac{\rho}{\rho-\xi}\frac{1}{1-\alpha\xi}\Psi(r,\xi)d\xi=\frac{\rho}{1-\alpha\rho}\Psi(r,\rho)+\frac{\rho}{\alpha\rho-1}\Psi\left(r,\frac{1}{\alpha}\right), \tag{6.10}$$

since on behalf of (6.6) $\Phi(r,\xi)=0(1)$ for $|\xi|\to\infty$, $|\arg\xi|<\frac{1}{2}\pi$.

Since (6.10) holds for all $\operatorname{Re}\rho>0$, it follows for $|r|<1$,

$$\frac{r(1-\beta(\rho))}{\rho}\Phi\left(r,\rho+\frac{1}{c}\right)+\frac{1-r-\alpha\rho}{\rho}\Phi(r,\rho)-\frac{1-\alpha\rho}{\rho}r=C(r), \tag{6.11}$$

with $C(r)$ a function of r independent of ρ. From (6.11) we can determine $\Phi(r,\rho)$ and hence obtain the time dependent solution. However, we shall restrict the discussion to the stationary solution of the problem. On behalf of the results of Afanas'eva and Martynov [2] this stationary solution exists. The method which we shall use to obtain the stationary solution can be also applied for the determination of the solution of (6.11). Once this latter solution has been obtained the existence of the stationary solution can be derived from it by applying the usual techniques of renewal theory.

Since the stationary solution exists the following limits exist

$$\Phi(\rho)\overset{\text{def}}{=}\lim_{n\to\infty}E\{\exp(-\rho w_n)-w_1=0\}=\lim_{r\uparrow 1}(1-r)\Phi(r,\rho),\ \operatorname{Re}\rho\geqq 0, \tag{6.12}$$

$$C\overset{\text{def}}{=}\lim_{r\uparrow 1}(1-r)C(r). \tag{6.13}$$

Consequently, from (6.11) for $\operatorname{Re}\rho\geqq 0$,

$$\frac{1-\beta(\rho)}{\rho}\Phi\left(\rho+\frac{1}{c}\right)-\alpha\Phi(\rho)=C. \tag{6.14}$$

With

$$D\overset{\text{def}}{=}-C/\alpha$$

it follows from (6.14) for $\operatorname{Re}\rho\geqq 0$,

$$\begin{aligned}\Phi(\rho)&=\frac{1-\beta(\rho)}{\alpha\rho}\Phi\left(\rho+\frac{1}{c}\right)+D\\&=D+\frac{1-\beta(\rho)}{\alpha\rho}D+\frac{1-\beta(\rho)}{\alpha\rho}\frac{1-\beta\left(\rho+\frac{1}{c}\right)}{\left(\rho+\frac{1}{c}\right)\alpha}\Phi\left(\rho+\frac{2}{c}\right)\end{aligned} \tag{6.15}$$

$$= D\left\{1 + \frac{1-\beta(\rho)}{\alpha\rho} + \frac{1-\beta(\rho)}{\alpha\rho}\frac{1-\beta\left(\rho+\frac{1}{c}\right)}{\left(\rho+\frac{1}{c}\right)\alpha}\right.$$

$$\left. + \frac{1-\beta(\rho)}{\alpha\rho}\frac{1-\beta\left(\rho+\frac{1}{c}\right)}{\left(\rho+\frac{1}{c}\right)}\frac{1-\beta\left(\rho+\frac{2}{c}\right)}{\left(\rho+\frac{2}{c}\right)}\Phi\left(\rho+\frac{3}{c}\right)\right\},$$

and so on.

Since

$$\lim_{n\to\infty} \Pr\{w_n=0 \mid w_1=0\} = \lim_{|\rho|\to\infty} \Phi(\rho), \qquad |\arg\rho| < \tfrac{1}{2}\pi, \tag{6.16}$$

it follows from (6.14) for ρ real

$$\lim_{\rho\to\infty} \Phi(\rho) = D, \tag{6.17}$$

and hence from (6.15) for Re $\rho \geqq 0$,

$$\Phi(\rho) = D\left\{1 + \sum_{i=0}^{\infty}\prod_{j=0}^{i}\frac{1-\beta\left(\rho+\frac{j}{c}\right)}{\left(\rho+\frac{j}{c}\right)}\right\}. \tag{6.18}$$

The norming condition yields

$$D^{-1} = 1 + \sum_{i=0}^{\infty}\prod_{j=0}^{i}\frac{1-\beta\left(\frac{j}{c}\right)}{j\frac{\alpha}{c}}, \tag{6.19}$$

with

$$\frac{1-\beta\left(\frac{j}{c}\right)}{j\frac{\alpha}{c}} \overset{\text{def}}{=} \beta/\alpha = a \quad \text{for} \quad j=0. \tag{6.20}$$

The probability of congestion, i.e. the probability that an arriving customer is not admitted to the system, is given by

$$\begin{aligned}\lim_{n\to\infty} \Pr\{w_n > u_n \mid w_1 = 0\} &= \int_{0-}^{\infty} \{1-e^{-w/c}\}\, d_w \lim_{n\to\infty} \Pr\{w_n < w \mid w_1 = 0\} \\ &= 1 - \Phi\left(\frac{1}{c}\right) \\ &= 1 - D\left\{1 + \sum_{i=0}^{\infty}\prod_{j=0}^{i}\frac{1-\beta((j+1)/c)}{(j+1)\alpha/c}\right\}.\end{aligned} \tag{6.21}$$

Next, we particularize the result obtained above for

$$\beta(\rho) = \frac{1}{1+\beta\rho}, \qquad \beta > 0. \tag{6.22}$$

We obtain from (6.18) and (6.22) for Re $\rho \geqq 0$,

$$\Phi(\rho) = D\left\{1 + \sum_{i=0}^{\infty} \prod_{j=0}^{i} \frac{ac/\beta}{(1+\rho\beta)c/\beta+j}\right\}$$

$$= D\left\{1 + \sum_{i=0}^{\infty} \left(\frac{ac}{\beta}\right)^{i+1} \frac{\Gamma\left(\frac{c}{\beta}(1+\rho\beta)\right)}{\Gamma\left(i+1+(1+\rho\beta)\frac{c}{\beta}\right)}\right\}. \qquad \text{Re}\,\rho \geqq 0\,.$$

We now have (cf. Erdélyi, [3], vol. 1),

$$\frac{\Gamma(\rho)}{\Gamma(\rho+j)} = \int_0^{\infty} e^{-\rho t} \frac{(1-e^{-t})^{j-1}}{\Gamma(j)}\, dt \quad \text{for} \quad j>0\,.$$

It follows

$$\Phi(\rho) = D\left\{1 + \frac{ac}{\beta}\int_0^{\infty} \exp\left\{-\frac{c}{\beta}(1+\beta\rho)t + \frac{ac}{\beta}(1-e^{-t})\right\} dt\right\}, \qquad \text{Re}\,\rho \geqq 0\,. \tag{6.23}$$

Denoting by $W(t)$ the stationary distribution of the waiting time, so that

$$W(t) = \lim_{n\to\infty} \Pr\{w_n < t \mid w_1 = 0\}$$

it follows from (6.23),

$$W(t) = 0\,, \qquad t<0\,,$$

$$= \frac{1+a\int_0^{t/\beta} \exp\left\{-u+(1-e^{-u\beta/c})\frac{ac}{\beta}\right\} du}{1+a\int_0^{\infty} \exp\left\{-u+(1-e^{-u\beta/c})\frac{ac}{\beta}\right\} du}\,, \quad t>0\,. \tag{6.24}$$

In particular if $\beta=c$ we obtain from (6.24)

$$W(\beta t) = D\left\{1+a\int_0^{t} e^{-u+a(1-e^{-u})}\, du\right\}$$

$$= D\left\{1-a\int_1^{e^{-t}} e^{a(1-s)}\, ds\right\}$$

$$= D\, e^{a(1-e^{-t})}, \qquad t>0\,,$$

so that

$$D = e^{-a}, \quad W(t) = e^{-ae^{-(t/\beta)}}, \qquad t>0\,. \tag{6.25}$$

For the present case the probability of congestion is given by (cf. (6.21)),

$$\{e^{-a} - (1-a)\}\, a^{-1}\,. \tag{6.26}$$

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